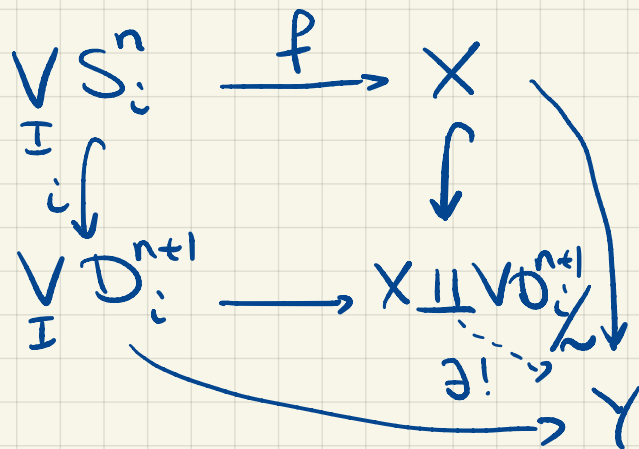


Homotopy Theory

25 October

① CW-approximation



Obj: 0, 1, 2

$\mathcal{P}_0: 2 \leftarrow 0 \rightarrow 1$

$F: \mathcal{P}_0 \rightarrow \text{Top}$

$\text{colim}_{\mathcal{P}_0} F \in \text{Top}$

$s \in S_i^n, f(s) \sim i(s)$

$$X \amalg (V D_i^{n+1}) / \sim = X \cup_f (U e_i^{n+1})$$

Isolate one step

X is a path-connected space

$x_0 \in X$ base point

Let $n \geq 2$. Then $\pi_n X = A$ is an abelian group

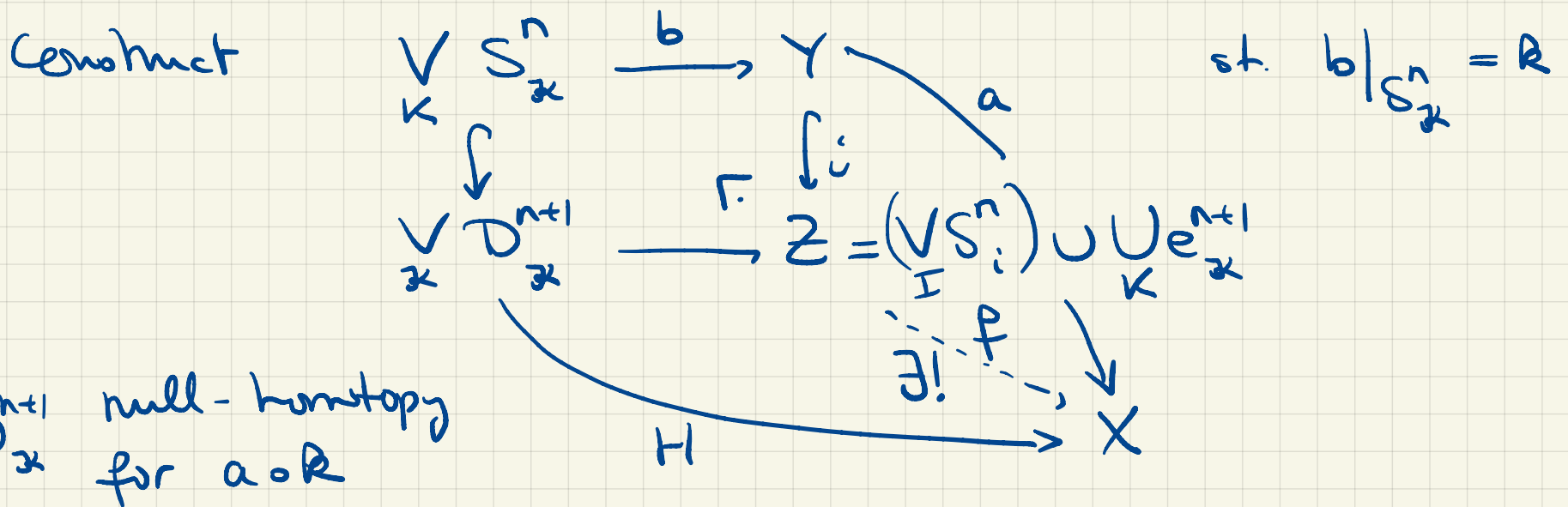
(part 1:) construct a CW-complex Y & map $Y \rightarrow X$ s.t.
 $\pi_n Y \twoheadrightarrow \pi_n X = A$ surj.
 choose generators for A , α_i , choose representatives $S^n \xrightarrow{a_i} X$

assemble them to construct $a: \bigvee_i S_i^n \longrightarrow X$

(defined by the universal property of the wedge: $a|_{S_i^n} = a_i$).

$a_* = \pi_n(a)$ surjective because $z_i: S_i^n \hookrightarrow \bigvee_i S_i^n$ represents $[z_i] \in \pi_n(\bigvee_i S_i^n) \cong \bigoplus_i \mathbb{Z}$ freely generated as an ab. grp by $[z_i]$ s.t. $a_*[z_i] = [a \circ z_i] = [a_i] = \alpha_i$

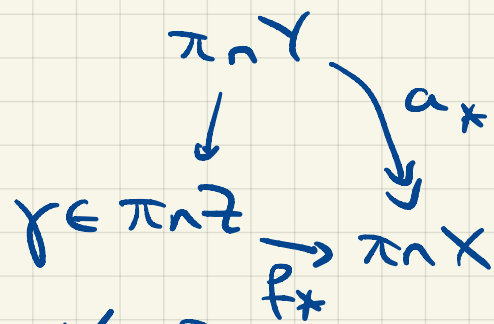
(part 2) Let $K = \text{Ker}(\pi_n Y \longrightarrow \pi_n X) \cong \text{Ker}(\bigoplus_i \mathbb{Z} \longrightarrow A)$
 For any $x \in K$, $x \in \pi_n Y$, representative $R: S^n \longrightarrow Y$ (so $S^n \xrightarrow{R} Y \xrightarrow{a} X$ is $\simeq *$)



$H|_{D_x^{n+1}}$ null-homotopy for $a \circ R$

Claim: $f_* = \pi_n(f)$ is an iso.

f_* surjective ✓

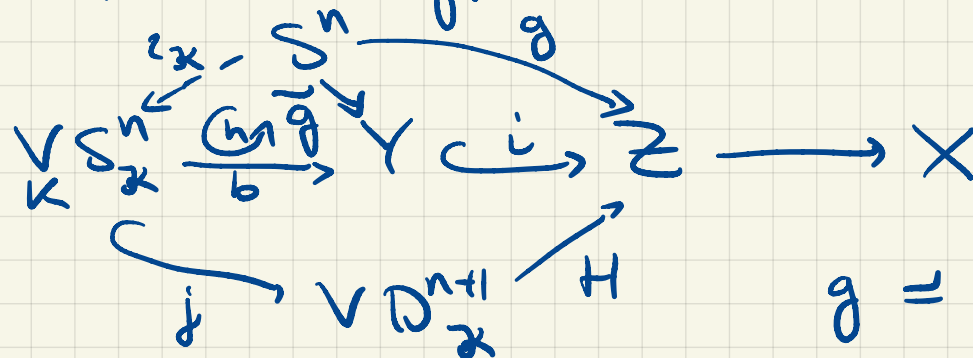


f_* injective:

Let $\gamma \in \text{Ker } f_*$, represent by $g: S^n \rightarrow Z$. By the cellular approximation Theorem, $g = g'$ cellular, so $g': S^n \xrightarrow{\tilde{g}} Z^{(n)} = \bigvee_I S_i^n = Y \hookrightarrow Z$

$$\begin{aligned} \text{Then } 0 &= f_*(\gamma) = [f \circ g] = [f \circ g'] = [\underbrace{f \circ i}_{=} \circ \tilde{g}] \\ &= [a \circ \tilde{g}] = a_*[\tilde{g}] \end{aligned}$$

So $[\tilde{g}] \in K$, so $\tilde{g} \simeq R$, up to homotopy we have a commutative diagram:



$$g = *$$

□

$$g \simeq i \circ \tilde{g} \simeq i \circ b \circ R = H \circ j \circ R \simeq c_{z_0}$$

Aim: construct a space X with $\pi_n X \cong \mathbb{Z}/p^k$

$$X^{(n)} = S^n$$

$$\pi_n S^n \cong \mathbb{Z} \text{ surjects on } \mathbb{Z}/p^k$$

Call $p_k: \mathbb{Z} \rightarrow \mathbb{Z}/p^k$.

$$X = S^n \cup_{p_k} e^{n+1}$$

p^k is $p^k \cdot \text{id}_{S^n}: S^n \xrightarrow{p^k} \bigvee_{p^k} S^n \xrightarrow{\Delta} S^n$
 It's k map of degree p^k

$$\begin{array}{ccc} S^n & \xrightarrow{p^k} & S^n \\ \downarrow & \lrcorner & \downarrow \\ D^{n+1} & \longrightarrow & S^n \cup_{p^k} e^{n+1} \end{array}$$

Lemma. The map of degree $a \in \mathbb{N}$ on S^n induces multiplication by a on $H_n(-; \mathbb{Z})$

$$a_*: H_n(S^n; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\cdot a} H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$$

$a=2$:

$$S^n \xrightarrow{p^2} S^n \vee S^n \xrightarrow{\Delta} S^n$$

degree 2?

$$\begin{array}{ccccc} H_n(-; \mathbb{Z}) & \mathbb{Z} & \xrightarrow{\Delta} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{+} & \mathbb{Z} \\ & 1 & \longmapsto & & & 2 \end{array}$$

Claim: $\pi_n X \cong \mathbb{Z}/p^k$

① $\pi_n X$ is a quotient of $\mathbb{Z}$ (X, S^n)

l.e.s in π_n : $\mathbb{Z} \cong \pi_n S^n \xrightarrow{i_*} \pi_n X \rightarrow \underbrace{\pi_n(X, S^n)}_{\cong 0} \xrightarrow{\cong}$

$$\cong [(\mathbb{D}^n, S^{n-1}), (X, S^n)] = 0$$

by cellular approx. (relative)

$$(\mathbb{D}^n, S^{n-1}) \rightarrow (S^n, S^n) \rightarrow (X, S^n)$$

$$i: S^n \hookrightarrow X$$

② $i_*(p^k) = 0$.

$$\parallel$$

$$[i \circ p^k] = 0$$

$$\begin{array}{ccc} S^n & \xrightarrow{p^k} & S^n \\ \downarrow & & \downarrow i \\ \mathbb{D}^{n+1} & \xrightarrow{\quad} & X \end{array} \quad \text{commutes}$$

since $i \circ p^k$ factors through $\mathbb{D}^{n+1} = *$.

So $\pi_n X$ is a quotient of $\mathbb{Z}/p^k$

③ $\pi_n X \cong \mathbb{Z}/p^k$. $\Leftrightarrow p^{k-1} \neq 0$ in $\pi_n X$

$(\Rightarrow) S^n \xrightarrow{p^{k-1}} S^n \xrightarrow{i} X$ is not null-homotopic

This will be so if it's not zero in $H_n(-; \mathbb{Z})$

$$\begin{array}{ccccc}
 H_n(S^n; \mathbb{Z}) & \xrightarrow{(p^k-1)_*} & H_n(S^n; \mathbb{Z}) & \xrightarrow{i_*} & H_n(X; \mathbb{Z}) \\
 \cong & & \cong & & \cong \\
 \mathbb{Z} & \xrightarrow{\cdot p^{k-1}} & \mathbb{Z} & \xrightarrow{p^k} & \mathbb{Z}/p^k
 \end{array}$$

$\underbrace{\hspace{15em}}_{\text{Lemma}} \neq 0$

$$H_n(X; \mathbb{Z}) \cong H_n^{\text{cell}}(X; \mathbb{Z}) \stackrel{\text{def}}{=} H_n(C_*^{\text{cell}}(X; \mathbb{Z}))$$

$$\begin{array}{ccccccccccccccc}
 & & n+2 & & n+1 & & n & & n-1 & & n-2 & & \dots & & 1 & & 0 \\
 C_*^{\text{cell}}(X; \mathbb{Z}) & 0 \rightarrow & 0 & \rightarrow & \mathbb{Z} & \xrightarrow{d} & \mathbb{Z} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \dots & \rightarrow & 0 & \rightarrow & \mathbb{Z} & \rightarrow & 0
 \end{array}$$

d is determined by the attaching map, it's multiplication by its degree (p^k)

$$\left. \begin{array}{l}
 H_0(X; \mathbb{Z}) \cong \mathbb{Z} \quad (\text{it's connected}) \\
 H_n(X; \mathbb{Z}) \cong \mathbb{Z}/\text{Im } d = \mathbb{Z}/p^k \\
 H_m(X; \mathbb{Z}) = 0 \quad \forall m \neq 0, n
 \end{array} \right\} \begin{array}{l}
 X = \mathbb{M}(\mathbb{Z}/p^k, n) \\
 \text{Moore space} \\
 (\text{single non trivial reduced homology grp})
 \end{array}$$

Ex. 3 (4) $\Rightarrow$ (1) Hint. Show $K \rightarrow X$ factors through

$K/K^{(m)}$ up to homotopy

Induction on m , like cellular approximation

(Please use the HEP without reproving it!)