Homotopy Theory

25 October

1 CW-approximation

$$\begin{tikzcd}
\bigvee_I S^n_i \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \arrow[ddr, bend left] & \\
\bigvee_I D^{n+1}_i \arrow[r] \arrow[drr, bend right] & X \amalg \bigvee D^{n+1}_i/\sim \arrow[dr, dashed, "\exists!"] & \\
 & & Y
\end{tikzcd}$$

Obj : 0, 1, 2

$P_0 : 2 \longleftarrow 0 \longrightarrow 1$

$F : P_0 \longrightarrow \mathrm{Top}$

$\mathrm{colim}_{P_0} F \in \mathrm{Top}$

$s \in S^n_i$, $f(s) \sim i(s)$

$$X \amalg \left(\bigvee D^{n+1}_i\right)/\sim \; = X \cup_f \left(\bigcup_i e^{n+1}_i\right)$$

Isolate one step

X is a path-connected space

$x_0 \in X$ base point

Let $n \geq 2$. Then $\pi_n X = A$ is an abelian group

part 1: construct a CW-cx Y & map $Y \longrightarrow X$ s.t.

$\pi_n Y \twoheadrightarrow \pi_n X = A$ surj

choose generators for A, α_i, choose representatives $S^n \xrightarrow{a_i} X$

assemble them to construct $a : \bigvee_i S^n_i \longrightarrow X$

(defined by the universal property of the wedge: $a|_{S^n_i} = a_i$).

$a_* = \pi_n(a)$ surjective because $\iota_i : S^n_i \hookrightarrow \bigvee_I S^n_i$ represents

$[\iota_i] \in \underbrace{\pi_n(\bigvee S^n_i)}_{\cong \bigoplus_I \mathbb{Z}}$ s.t. $a_*[\iota_i] = [a \circ \iota_i] = [a_i] = \alpha_i$

freely generated as an ab. grp by $[\iota_i]$

part 2 Let $K = \ker(\pi_n Y \longrightarrow \pi_n X) \cong \ker(\bigoplus_I \mathbb{Z} \longrightarrow A)$

For any $\kappa \in K$, $\kappa \in \pi_n Y$, representative $k : S^n \longrightarrow Y$ (so $S^n \xrightarrow{k} Y \xrightarrow{a} X$ is $\simeq *$)

construct

$$\begin{tikzcd}
\bigvee_K S^n_\kappa \arrow[r, "b"] \arrow[d, hook] & Y \arrow[d, "i"] \arrow[ddr, bend left, "a"] & \\
\bigvee_\kappa D^{n+1}_\kappa \arrow[r] \arrow[drr, bend right, "H"'] & Z = (\bigvee_I S^n_i) \cup \bigcup_K e^{n+1}_\kappa \arrow[dr, dashed, "\exists!\, f"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & X
\end{tikzcd}$$

st. $b|_{S^n_\kappa} = k$

$H|_{D^{n+1}_\kappa}$ null-homotopy for $a \circ k$

Claim: $f_* = \pi_n(f)$ is an iso.

$$\begin{tikzcd} & \pi_n Y \arrow[d] \arrow[dr, bend left, "a_*"] & \\ & \gamma \in \pi_n Z \arrow[r, "f_*"'] & \pi_n X \end{tikzcd}$$

f_* surjective ✓

f_* injective: Let $\gamma \in \operatorname{Ker} f_*$, represent by $g : S^n \longrightarrow Z$. By the cellular approximation Theorem, $g \simeq g'$ cellular, so $g' : S^n \xrightarrow{\tilde{g}} Z^{(n)} = \bigvee_I S^n_i = Y \overset{i}{\subset} Z$

Then $0 = f_*(\gamma) = [f \circ g] = [f \circ g'] = [\underbrace{f \circ i} \circ \tilde{g}]$

$ = [a \circ \tilde{g}] = a_*[\tilde{g}]$

So $\underset{k}{\overset{\|}{[\tilde{g}]}} \in K$, so $\boxed{\tilde{g} \simeq k}$, up to homotopy we have a commutative diagram:

$$\begin{tikzcd} & S^n \arrow[dl, "\iota_x"'] \arrow[d, "\tilde{g}"] \arrow[dr, bend left, "g"] & & \\ \bigvee_K S^n_x \arrow[r, "b"'] \arrow[dr, hook, "j"'] & Y \arrow[r, hook, "i"] & Z \arrow[r] & X \\ & \bigvee D^{n+1}_x \arrow[ur, "H"'] & & \end{tikzcd}$$

$g \simeq *$. □

$g \simeq i \circ \tilde{g} \simeq i \circ b \circ k = H \circ j \circ k \simeq c_{z_0}$.

Aim: construct a space X with $\pi_n X \cong \mathbb{Z}/p^k$

$X^{(n)} = S^n$ $\qquad$ $\pi_n S^n \cong \mathbb{Z}$ surjects on $\mathbb{Z}/p^k$

Call $p_k : \mathbb{Z} \longrightarrow \mathbb{Z}/p^k$.

$$\boxed{X = S^n \cup_{p^k} e^{n+1}}$$

$$\begin{tikzcd} S^n \arrow[r, "p^k"] \arrow[d, hook] & S^n \arrow[d, hook] \\ D^{n+1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^n \cup_{p^k} e^{n+1} \end{tikzcd}$$

p^k is $p^k \cdot \mathrm{id}_{S^n} : S^n \xrightarrow{\text{pinch}} \bigvee_{p^k} S^n \xrightarrow{\nabla} S^n$

It's the map of degree p^k

Lemma. The map of degree $a \in \mathbb{N}$ on S^n induces multiplication by a on $H_n(-;\mathbb{Z})$

$$a_* : H_n(S^n;\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\cdot a} H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$$

$a=2$: $\quad S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\nabla} S^n \qquad$ degree 2?

$H_n(-;\mathbb{Z}) \quad \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{+} \mathbb{Z}$

$\qquad\qquad\qquad\quad 1 \longmapsto 2$

Claim: $\pi_n X \cong \mathbb{Z}/p^k$

① $\pi_n X$ is a quotient of $\mathbb{Z}$ $\qquad (X, S^n)$

l.e.s in π_*: $\mathbb{Z} \cong \pi_n S^n \xrightarrow{i_*} \pi_n X \longrightarrow \underbrace{\pi_n(X, S^n)}_{\cong [(D^n, S^{n-1}), (X, S^n)] = 0} \xrightarrow{\partial}$

$i : S^n \hookrightarrow X$

by cellular approx. (relative)

$(D^n, S^{n-1}) \to (S^n, S^n) \to (X, S^n)$

② $i_*(p^k) = 0$.
$\quad \|$
$[i \circ p^k] = 0$

$$\begin{tikzcd} S^n \arrow[r, "p^k"] \arrow[d, hook] & S^n \arrow[d, hook, "i"] \\ D^{n+1} \arrow[r, "\Gamma"] & X \end{tikzcd}$$ commutes

since $i \circ p^k$ factors through $D^{n+1} \simeq *$.

So $\boxed{\pi_n X \text{ is a quotient of } \mathbb{Z}/p^k}$

③ $\pi_n X \cong \mathbb{Z}/p^k$. $\iff$ $p^{k-1} \neq 0$ in $\pi_n X$

$\iff$ $S^n \xrightarrow{p^{k-1}} S^n \xhookrightarrow{i} X$ is not null-homotopic

This will be so if it's not zero in $H_n(\_\,; \mathbb{Z})$

$$\begin{array}{ccccc} H_n(S^n;\mathbb{Z}) & \xrightarrow{(p^k-1)_*} & H_n(S^n;\mathbb{Z}) & \xrightarrow{i_*} & H_n(X;\mathbb{Z}) \\ \cong & & \cong & & \cong \\ \mathbb{Z} & \xrightarrow{\cdot p^{k-1}} & \mathbb{Z} & \xrightarrow{pr} & \mathbb{Z}/p^k \end{array}$$

Lemma

$\neq 0$

$$H_n(X;\mathbb{Z}) \cong H_n^{cell}(X;\mathbb{Z}) \overset{def}{=} H_n(C_*^{cell}(X;\mathbb{Z}))$$

$$\begin{array}{lccccccccccccc} & & & n+2 & & n+1 & & n & & n-1 & & n-2 & \cdots & 1 & & 0 \\ C_*^{cell}(X;\mathbb{Z}) & 0 \to & 0 \to & \mathbb{Z} & \xrightarrow{d} & \mathbb{Z} & \to & 0 & \to & 0 & \to & & \to 0 \to & \mathbb{Z} & \to & 0 \end{array}$$

one n-cell

d is determined by the attaching map, it's multiplication by its degree (p^k)

$$\left.\begin{array}{ll} H_0(X;\mathbb{Z}) \cong \mathbb{Z} & \text{(it's connected)} \\ H_n(X;\mathbb{Z}) \cong \mathbb{Z}/\mathrm{Im}\, d = \mathbb{Z}/p^k & \\ H_m(X;\mathbb{Z}) = 0 & \forall\, m \neq 0, n \end{array}\right\} \quad X = M(\mathbb{Z}/p^k, n)$$

Moore space (single non trivial reduced homology grp)

<u>Ex. 3</u> (4) $\Rightarrow$ (1) Hint. Show $K \longrightarrow X$ factors through

$K/K^{(n)}$ up to homotopy

Induction on n, like cellular approximation

Please use the HEP without reproving it!